Effective Wall Heat Transfer Coefficients and Thermal Resistances in Mathematical Models of Packed Beds

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Expressions are derived for an effective wall heat transfer coefficient useful in one-dimensional representations of heat transport in packed beds. These expressions are obtained with two mathematical models of a cylindrical packed bed: a partial differential model and a finite stage model. These expressions relate the effective wall heat transfer coefficient, which is a local coefficient, to the actual wall heat transfer coefficient and the bed diameter (and the radial Peclet number in the partial differential model) in regions of the bed where similar temperature profiles obtain. These relations involve a single dominant root of an equation characteristic of the heat balance equations of each model.

From these relations, expressions for an effective thermal resistance of the bed are obtained. For each model, this bed resistance is found to be an approximately linear function of the bed diameter and to be rather insensitive to the actual wall heat transfer coefficient. For each model, an approximate bed resistance is found that is not dependent upon the actual wall heat transfer coefficient and with which the effective wall heat transfer coefficient can be estimated with an error of less than 7%.

The rate of heat transfer through the wall of a cylindrical packed bed of solids through which fluid is flowing is often conveniently calculated with a one-dimensional representation. Such calculations use an effective wall heat transfer coefficient based upon some mean temperature difference. The effective heat transfer coefficient considered here is h_1, a local coefficient defined in terms of the local rate of heat flow δq through a wall area $c\delta x$

$$\delta q(x) = h_1\, c\delta x\, [\overline{\theta}(x) - \theta_e(x)] \qquad (1)$$

Equation (1) is the defining equation for h_1. The coefficient is artificial because the driving force, based on a radially averaged temperature, is artificial; experience has shown, however, that such a coefficient is practically useful. The coefficient h_1 may generally be expected to vary with axial position and should reflect both the thermal resistance at the tube wall and the radial thermal resistance of the packed bed itself.

The heat flow $\delta q(x)$ in Equation (1) may be determined from the relation

$$\delta q(x) = h_2\, c\delta x\, [\theta(x)_{\text{at wall}} - \theta_e(x)] \qquad (2)$$

Although Equation (2) could properly be considered as the defining equation for h_2, instead it is treated here as the equation for the determination of δq. Because the driving force in Equation (2) is more justifiable physically than is the driving force in Equation (1), h_2 is assumed to be a fundamental parameter of the two-dimensional models considered here; also, h_2 is assumed to be known and independent of x. The effective local heat transfer coefficient h_1 is thus a function of h_2 and the other properties of the bed through the relationship

$$h_1[\overline{\theta}(x) - \theta_e(x)] = h_2[\theta(x)_{\text{at wall}} - \theta_e(x)] \qquad (3)$$

One purpose of this study is to determine, starting from Equation (3), the form of the dependence of h_1 on h_2 and other bed parameters for two common mathematical models of a packed bed: the well-known partial differential model and the finite stage model (7). Design procedures and the phenomenology of heat transfer in packed beds are not discussed here, as they are amply treated elsewhere (1, 2, 3, 13). Thus, any dependence of h_1 on the Reynolds number or the Prandtl number is not considered here, for such dependence is implicit through h_2 and other bed parameters. This study only investigates certain consequences of some common definitions and concepts concerning heat transfer in packed beds.

Considered here are two common cases in which heat is transferred radially in a packed bed. The first case is that in which a heated fluid, passing through a bed, loses heat to the exterior medium which is maintained at a constant temperature. The second case is that in which the flux of heat through the tube wall is constant and the temperature of the flowing fluid continually increases down the bed. The first case is treated in some detail here. Although Quinton and Storrow (12) have shown the second case to have some experimental advantages in the determination of heat transfer coefficients, this second case is shown here to be, in some respects, just a limiting form of the first case.

For the first case, with constant exterior temperature θ_e, the analysis is facilitated by the use of dimensionless variables and parameters. With normalized temperatures

$$T(x, r) \equiv \frac{\theta(x, r) - \theta_e}{\theta_o - \theta_e}$$

and dimensionless heat transfer coefficients

$$H_1 \equiv h_1/Gc_p, \quad H_2 \equiv h_2/Gc_p$$

Equation (3) becomes

$$H_1\overline{T}(x) = H_2 T_w(x) \qquad (4)$$

Here, the temperature θ_o of the entering fluid is assumed to be uniform across the radius.

Although the effective heat transfer coefficient H_1 generally may be dependent upon axial position, H_1 easily is shown to be independent of x for regions of the bed in which temperature profiles are similar; that is, for regions in which temperature can be represented by

$$T(x, r) = \tau_x(x) \cdot \tau_r(r)$$

where τ_x is a function of x only and τ_r is a function of r only. Similar temperature profiles are not an assumption in this derivation; rather, the solutions of heat balance equations show that temperature profiles are approximately similar beginning at about three or four tube diameters from the inlet of the packed bed. The derivation of the behavior of H_1 applies to these regions of temperature similarity.

The total resistance to heat flow in the radial direction $1/H_1$ may be considered as the sum of two thermal resistances: the resistance to heat flow at the wall $(1/H_2)$ and a resistance to heat flow radially through the bed.

$$\frac{1}{H_1} = R + \frac{1}{H_2} \qquad (5)$$

This equation is the defining equation for R, the local radial thermal resistance of the bed, hereafter called the bed resistance. The concept of this bed resistance is artificial because it is associated with an artificial driving force $[\bar{\theta}(x) - \theta(x)_{\text{at wall}}]$ and the bed resistance generally depends upon H_2. The present analysis shows, however, that R is not strongly dependent upon H_2, and that R can be satisfactorily replaced with an approximate value that is not dependent on H_2.

The present relations for H_1 and R are developed partly to estimate the local rates of heat loss with only one-dimensional representations of packed beds. Determination of local rates of heat loss are particularly important for packed bed chemical reactors. Similar temperature profiles are not generally found in such reactors, however, or in packed beds in unsteady state operation. But one-dimensional representations can certainly be used in such cases if it can be shown that such representations satisfactorily describe the behavior of the packed bed. The authors used these heat transfer coefficient relations in a study of transients in chemical reactors (5, 6), where it was shown that one-dimensional models were satisfactory.

On the other hand, these expressions should not be used in design problems to estimate bed lengths or terminal temperatures, for example; such problems are more properly treated with an overall heat transfer coefficient (13) which is a function of tube length and which allows for the entrance effects that have been disregarded here.

The first of the two mathematical models of a packed bed treated here is the familiar differential model in which the bed is treated as a continuum. Radial and axial heat transport through the bed is treated in this model as an effective conduction process, with constant but distinct effective thermal conductivities in the radial and axial directions. For this model, the heat balance is in the form of a familiar partial differential equation.

The second model considered here is the finite stage model proposed by Deans and Lapidus (7), in which the packed bed is represented by a two-dimensional array of discrete stages. Figure 1 shows schematically such an array of finite stages and the motion of fluid among them. These stages are perfectly stirred tanks and are ring shaped with square cross section. The length of the cross section is equal to the average diameter of a packing particle d_p. Each stage represents a void space between a group of contiguous particles in which turbulent fluid mixing occurs. Heat is transported from stage to stage by fluid motion only; thermal conduction both in the fluid and in the solid is neglected. For this model the heat balance is in the form of a difference equation. These equations may be conveniently expressed in a matrix form.

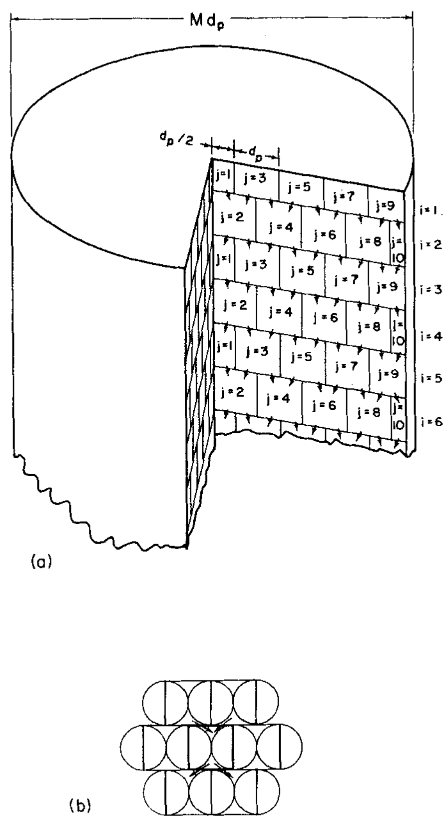

Fig. 1. Arrangement of stages in the finite stage model. (*a*) Details of stage arrangement; here, $M = 9$. (*b*) Arrangement of particles upon which this model is based.

The development of the expression for H_1 through Equation (4) is essentially the same for each of the two models treated here. Solution of the heat balance appropriate for each model reveals that H_1 is determined by a single root of an equation characteristic of the heat balance. In the differential model, the characteristic equation is a nonlinear equation that involves Bessel functions and follows from the radial boundary conditions common to cylindrically symmetric heat conduction problems. For the finite stage model, the characteristic equation is the characteristic polynomial of a matrix that represents the stage heat balances. Determination of H_1 for each model is thus reduced to the well-known problem of finding roots. In the resulting relations for each model, H_1 is expressed as an explicit function of H_2 and a dimensionless tube diameter M; the expression in the differential representation includes an additional parameter, the radial Peclet number Pe_r. At small values of H_2, the relation for the differential model reduces to Beek's correlation (2), which holds for parabolic temperature profiles. Since the present study establishes a more general functional relation, the results presented here should serve as a useful extension of Beek's work.

THE PARTIAL DIFFERENTIAL MODEL

Values of the average temperature $\overline{T}(x)$ and the fluid temperature at the wall $T_w(x)$ required in Equation (4) may be obtained from the solution of the heat balance for a differential element of the cylindrical bed. For this case, the steady state heat balance is

$$\frac{1}{Pe_x}\frac{\partial^2 T}{\partial x^2}+\frac{1}{r}\frac{1}{Pe_r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)-\frac{\partial T}{\partial x}=0 \quad (6)$$

A radially uniform fluid velocity and constancy of fluid properties and effective thermal conductivities are assumed here. The dimensionless variables x and r express distances in terms of d_p.

The following analysis is similar to that given by Kjaer *(10)*. The radial boundary conditions are

1. Temperature symmetry at the axis, $r = 0$:

$$\frac{\partial}{\partial r}[T(x,r)]_{r=0}=0 \quad (7)$$

2. Continuity of heat flux at the wall, $r = M/2$:

$$\frac{1}{Pe_r}\frac{\partial}{\partial r}[T(x,r)]_{r=M/2}+H_2T(x,M/2)=0 \quad (8)$$

Here $T(x, M/2)$ is $T_w(x)$, the temperature of the fluid at the wall.

It can be shown easily that the axial boundary conditions are unimportant to the correlation derived here as long as approximately similar radial temperature profiles are obtained in the bed; this correlation depends only on the radial boundary conditions and the similarity of temperature profiles. Therefore, for convenience, the packed bed is assumed to be a semi-infinite bed, in which case a boundary condition is

3.

$$\lim_{x\to\infty} T(x,r)=0 \quad (9)$$

And the inlet boundary condition is taken as *(14)*

4.

$$T(0,r)-\frac{1}{Pe_x}\frac{\partial T}{\partial x}(0,r)=1 \quad (10)$$

The solution of Equation (6), subject to these four conditions, is

$$T(x,r)=\sum_{n=1}^{\infty}\frac{2\,Bi\,J_0\left(\frac{2r}{M}\beta_n\right)\exp(\rho_n x)}{(Bi^2+\beta_n^2)(1-\rho_n/Pe_x)J_1(\beta_n)} \quad (11)$$

where ρ_n is the negative root of $\rho^2 - Pe_x\rho - 4Pe_x\beta_n/M^2Pe_r = 0$, β_n is defined as the n^{th} positive root of

$$\beta J_1(\beta)-Bi\,J_0(\beta)=0 \quad (12)$$

and Bi is the Biot number defined by

$$Bi \equiv \tfrac{1}{2}MPe_rH_2=\tfrac{1}{2}D_Th_2/k_r \quad (13)$$

The Biot number is a measure of the ratio of the thermal resistance of the bed to the thermal resistance at the wall. It is seen that the terms β_n are functions only of the dimensionless parameter Bi.

Substitution of numerical values *(4)* of the roots β_n into Equation (11) shows that, except for a region near the inlet, only the first term in Equation (11) is important. Therefore, throughout most of the bed the fluid temperature can be represented by

$$T(x,r)=\frac{2\,Bi\,J_0\left(\frac{2r}{M}\beta_1\right)\exp(\rho_1 x)}{(Bi^2+\beta_1^2)(1-\rho_1/Pe_x)J_1(\beta_1)} \quad (14)$$

where β_1 is the smallest positive root of Equation (12). In this region the radial temperature profiles are similar, according to Equation (14).

The radially averaged temperature is

$$\overline{T}(x)\equiv\frac{8}{M^2}\int_0^{M/2}T(x,r)r\,dr=\frac{4Bi^2\exp(\rho_1 x)}{(Bi^2+\beta_1^2)(1-\rho_1/Pe_x)\beta_1^2} \quad (15)$$

Substitution of the expressions for $\overline{T}(x)$ and $T(x, M/2)$ in these last two equations into Equation (4) yields

$$H_1=\frac{[\beta_1(Bi)]^2}{MPe_r} \quad (16)$$

This equation, the goal of the preceding derivation, expresses H_1 as an explicit function of H_2 and other bed parameters. The function $[\beta_1(Bi)]^2$ is shown graphically in Figure 2. As Bi increases, β_1^2 approaches a limiting value, $\beta_{1\infty}^2 = 5.783186\ldots$, the square of the smallest positive root of J_0. For large Bi the effective heat transfer coefficient H_1 is influenced primarily by the thermal resistance of the bed, which, in this case, is much greater than the thermal resistance at the wall.

At the other extreme (low Bi), the effective heat transfer coefficient H_1 is approximately equal to the wall heat transfer coefficient H_2. As Bi becomes very small, H_1 approaches H_2, and β_1^2 approaches the 45-deg. asymptote shown in Figure 2. For small Bi the Bessel functions of Equation (12) may be approximated with a truncated Taylor's series:

$$\left(\frac{1}{2}\beta_1^2-\frac{1}{16}\beta_1^4\right)-Bi\left(1-\frac{1}{4}\beta_1^2+\frac{1}{64}\beta_1^4\right)\cong 0 \quad (17)$$

The smallest root of this equation is

$$\beta_1^2=\frac{8Bi}{Bi+4}=\frac{2Bi}{1+Bi/4} \quad (18)$$

With this approximation, Equation (16) yields

$$\frac{1}{H_1}=\frac{1}{H_2}+\frac{MPe_r}{8}$$

or, in dimensional terms

$$\frac{1}{h_1}=\frac{1}{h_2}+\frac{D_T}{8k_r} \quad (19)$$

These equations express Beek's correlation *(2)*. Beek derived these expressions by approximating radial tempera-

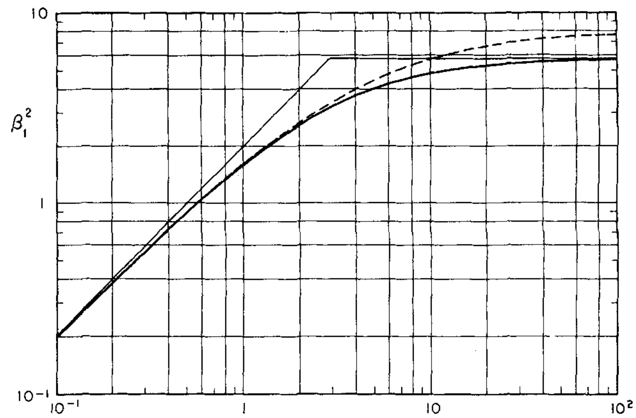

Fig. 2. The function $[\beta_1(Bi)]^2$; β_1 is the smallest positive root of the equation $\beta\ J_1(\beta)\ -\ Bi\ J_0(\beta)\ =\ 0$. ——— $= [\beta_1(Bi)]^2$. ——— = asymptotes. - - - - - = Beek's correlation, $2Bi/(1 + Bi/4)$.

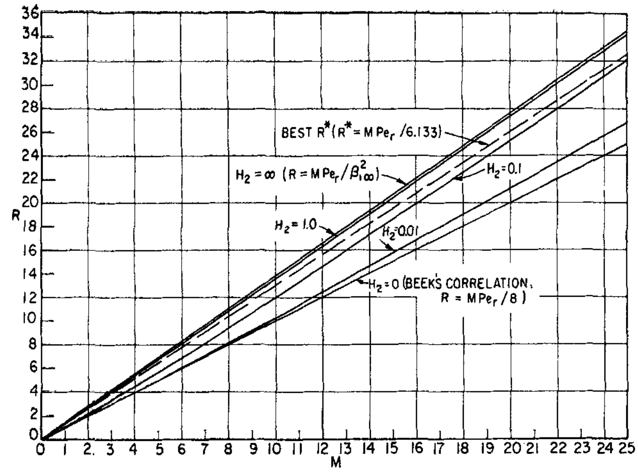

Fig. 3. The bed resistance R of the differential model as a function of M for selected values of H_2; here, $Pe_r = 8$.

ture profiles with a parabola, which is equivalent to approximating the Bessel functions with a truncated Taylor's series which is of second degree in β_1^2.

Beek's correlation, in the form of Equation (18), is shown graphically as a function of Biot number in Figure 2. It is accurate for small Bi, but for large Bi it is only qualitatively correct.

Comparison of Equations (19) and (5) shows that, for Beek's correlation

$$R = MPe_r/8 \text{ (for parabolic temperature profiles)} \quad (20)$$

For the packed bed in which the heat flux at the wall is constant along the length of the tube, Quinton and Storrow (*12*) have shown that the radial temperature profiles are parabolic. It is easily shown that when the radial temperature profiles are parabolic, Equation (20) results, regardless of how the packed bed is heated or cooled externally.

For the temperature distribution described by Equation (14), however, the expression for the thermal resistance is, from Equations (5) and (16)

$$R(M, Pe_r, H_2) = \frac{M\,Pe_r}{\left[\beta_1\left(\frac{1}{2} M\,Pe_r\,H_2\right)\right]^2} - \frac{1}{H_2} \quad (21)$$

Figure 3 shows the behavior of R calculated from Equation (21) as a function of M and H_2 for $Pe_r = 8$. This graph shows that R lies within a narrow triangular region bounded by the lines $R = MPe_r/8$ (Beek's correlation) and $R = MPe_r/\beta_{1\infty}^2$. As M increases for any constant value of H_2, R becomes a linear function of M, parallel to the line $R = MPe_r/\beta_{1\infty}^2$. The upper line represents the case in which there is no thermal resistance at the wall and the thermal resistance of the bed is controlling; the lower line (Beek's correlation) represents the limiting adiabatic case, or, from another viewpoint, the case in which the external heat flux is such that the radial temperature profiles are parabolic. Petersen (*11*) has shown that in many cases the thermal resistance of the bed is controlling; in such cases, the use of Equation (20) is inappropriate.

The fact that R is only slightly dependent upon H_2 suggests that the bed resistance may be treated as dependent only upon the properties of the fluid and the packing and independent of H_2. Such a bed resistance, called here R^*, is only an approximation to the "true" bed resistance R. With a known value of H_2 and the approximate bed resistance R^*, an approximate total resistance may be calculated.

$$\left(\frac{1}{H_1}\right)^* \equiv R^* + \frac{1}{H_2} \quad (22)$$

The appropriate value of R^* to use in Equation (22) is that value which makes $(1/H_1)^*$ the most accurate estimate of the total resistance $1/H_1$. The fractional error η for this estimate is

$$\eta = [(1/H_1)^* - (1/H_1)]/(1/H_1) \quad (23)$$

For the differential model several approximate bed resistances may be considered. The use of the bed resistance of Beek's correlation, Equation (20), leads to very low errors if the wall resistance is dominant; but if the bed resistance is controlling, the use of Beek's correlation leads to errors of up to 27.7%. The use of $R^* = MPe_r/\beta_{1\infty}^2$, the upper limit of R, leads to a maximum error (maximum value of η) of 8.7%. An analysis of Equation (16) reveals that the best approximate bed resistance for the differential model is

$$R^* = MPe_r/6.133 \quad (24)$$

The details of this analysis are presented in Appendix B.† This choice for R^* leads to a maximum error of only 5.7%, which is the lowest *maximum* error obtainable. This best bed resistance R^* that is independent of H_2 is shown as a function of M in Figure 3.

THE FINITE STAGE MODEL

The expression for H_1 for this model is derived in an analogous manner beginning with the heat balance in each stage. Heat balances are written for both the conjugate finite stage models shown in Figures 4 and 5. In model A (termed the even model), even numbered radial stages (j) are associated only with even numbered rows (i) and odd with odd. In its conjugate, model B (the odd model), odd numbered radial stages are associated with even rows and even with odd. Figures 4 and 5 display the stage arrangements and numbering convention for the conjugate models and show the location of the wall stages when M is even and when M is odd. It may be noted that the finite stage model shown in Figure 1 is only an even model.

The temperature profiles calculated with one model differ slightly from those of its conjugate. Since there is no reason to prefer one model to the other, a simple average of the conjugates is taken. Thus, the radially averaged temperature at row i is the average of the radially averaged temperatures $\overline{T}_i^A$ and $\overline{T}_i^B$ of the conjugates

$$\overline{T}_i = \frac{1}{2}[\overline{T}_i^A + \overline{T}_i^B] \quad (25)$$

A heat balance at a general stage (i, j) yields

$$(4j - 4)T_{i,j} = (2j - 3)T_{i-1,j-1} + (2j - 1)T_{i-1,j+1}$$
$$i = 1,2,\ldots;\ j = 2,3,\ldots, M - 1 \quad (26)$$

If i and j are both even or both odd, stage (i,j) is a stage of the even model, model A; if either i or j is even and the other is odd, stage (i,j) is a stage of the odd model, model B. The coefficients of the temperatures in this equation are proportional to the magnitude of the flows entering and leaving each stage. Since the fluid velocity profile is assumed uniform, these coefficients are proportional simply to the cross-sectional area common to two serial stages. Deans and Lapidus (7) show that the model may

† Deposited as document 8541 with the American Documentation Institute, Photoduplication Service, Library of Congress, Washington 25, D. C., and may be obtained for $1.25 for photoprints or 35-mm. microfilm.

easily be modified to account for radial variations in velocity.

Different balances follow for the center half-stage and the wall stages. For the center half-stage on any row i

$$T_{i,1} = T_{i-1,2} \tag{27}$$

For the wall stage that is a whole stage on row i

$$(4M - 4 + 4MH_2)T_{i,M} = (2M - 3)T_{i-1,M-1} + (2M - 1)_{i-1,M+1} \tag{28}$$

For the wall stage that is a half stage on row i

$$(2M - 1 + 4MH_2)T_{i,M+1} = (2M - 1)T_{i-1,M} \tag{29}$$

The terms $4MH_2T_{i,M}$ and $4MH_2T_{i,M+1}$ in Equations (28) and (29) represent the heat loss through the walls of these stages in accordance with Equation (2). As above, the fluid inlet temperature is taken as unity

$$T_{o,j} = 1 \qquad j = 1,2,\ldots, M+1 \tag{30}$$

For a given row i, Equations (26) through (29) can be expressed in a matrix form.

$$\mathbf{T}_i = \mathbf{A}\,\mathbf{T}_{i-1} \tag{31}$$

where

$$\mathbf{A} \equiv \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & & & \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 & \cdots & & 0 & \\ 0 & \frac{3}{8} & 0 & \frac{5}{8} & \cdots & & & \\ \cdots & & & & & & & \\ & \cdots & \frac{2j-3}{4j-4} & 0 & \frac{2j-1}{4j-4} & \cdots & & \\ \cdots & & & & & & & \\ \cdots & & & \frac{2M-3}{4M-4+4MH_2} & 0 & \frac{2M-1}{4M-4+4MH_2} & & \\ 0 & & & & & & & \\ \cdots & & 0 & \frac{2M-1}{2M-1+4MH_2} & 0 & & & \end{bmatrix} \tag{32}$$

and

$$\mathbf{T}_i \equiv \begin{bmatrix} T_{i,1} \\ T_{i,2} \\ \vdots \\ T_{i,j} \\ \vdots \\ T_{i,M} \\ T_{i,M+1} \end{bmatrix}, \qquad \mathbf{T}_o = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix} \tag{33}$$

$\mathbf{T}_i$ is the temperature vector; its elements are temperatures in all the radial stages of the conjugate models on row i. $\mathbf{T}_o$ is the vector which represents the temperature of the feed stream. The matrix of coefficients $\mathbf{A}$ represents the heat balance at any row of the composite model. $\mathbf{A}$ is a square matrix of order $M + 1$ with two interesting properties.

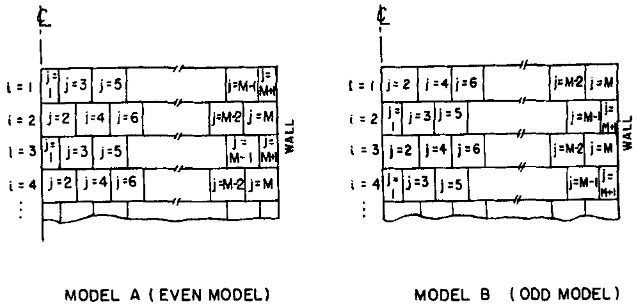

Fig. 4. Arrangement of stages in the conjugate finite stage models for a case in which M is even.

First, all elements of the matrix except those on the two diagonals adjacent to the main diagonal are zero. This property follows from the manner in which the stages of the model are arrayed. The zero principal and other diagonals of $\mathbf{A}$ insure the complete independence of the conjugate models; the temperature of a stage in one model will never affect the temperature in any stage of its conjugate.

Second, the sum of all elements on any row of the matrix, except the last two rows, is unity. This property expresses the conservation of heat in that part of the bed where temperature levels are affected by fluid mixing only. The last two rows are influenced by the loss of heat through the wall, and for the nonadiabatic case ($H_2 \neq 0$) the sum of the elements in these rows is less than unity. Because of this property, the absolute magnitudes of the roots of the characteristic polynomial of this matrix in the nonadiabatic case ($H_2 \neq 0$) are less than unity (9).

Equation (31) is a simple difference equation with constant coefficients. Its solution is

$$\mathbf{T}_i = \mathbf{A}^i\mathbf{T}_o \tag{34}$$

The matrix $\mathbf{A}^i$ is determined with Sylvester's theorem (8), which states that any polynomial function $P(\mathbf{A})$ of a square matrix $\mathbf{A}$ having distinct characteristic roots λ_r can be expressed as

$$P(\mathbf{A}) = \sum_{r=1}^{s} P(\lambda_r)\mathbf{J}(\lambda_r)/\varphi'(\lambda_r) \tag{35}$$

where s is the order of $\mathbf{A}$. Here, $P(\mathbf{A})$ is $\mathbf{A}^i$; thus

$$\mathbf{A}^i = \sum_{r=1}^{M+1} \lambda_r{}^i\,\mathbf{J}(\lambda_r)/\varphi'(\lambda_r) \tag{36}$$

and

$$\mathbf{T}_i = \sum_{r=1}^{M+1} \lambda_r{}^i\mathbf{J}(\lambda_r)\mathbf{T}_o/\varphi'(\lambda_r) \tag{37}$$

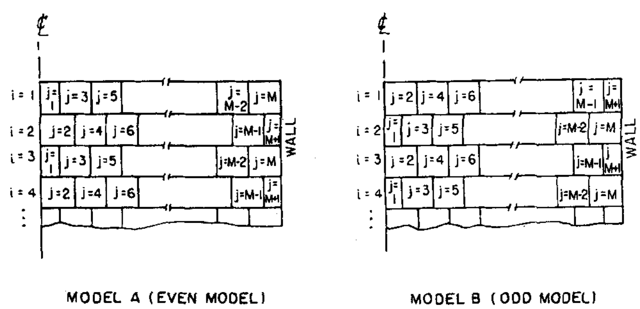

Fig. 5. Arrangement of stages in the conjugate finite stage models for a case in which M is odd.

This equation is the finite stage analog of Equation (11) for the differential representation. In both equations temperature is expressed as a series of exponential functions of axial distance, i or x. In both cases, in regions remote from the inlet of the bed (that is, i or x greater than $4M$), the only important term in each series is the first term, which includes the most slowly decreasing exponential function.

In the present case, however, λ_1, the largest characteristic root of **A**, is equal in absolute value to λ_2; actually, $\lambda_2 = -\lambda_1$. This results because $\varphi(\lambda)$ can be expressed as a polynomial in λ^2, owing to the first property of **A**. However, it can be shown by examples that the term containing λ_2 in Equation (37) is negligible because the elements of $\mathbf{J}(\lambda_2)$ are much smaller than the corresponding elements of $\mathbf{J}(\lambda_1)$. Therefore, in regions distant from the fluid entrance

$$\mathbf{T}_i = \lambda_1^{\,i}\,\mathbf{J}(\lambda_1)\mathbf{T}_o/\varphi'(\lambda_1) \tag{38}$$

Equation (38) is the finite stage analog of Equation (14) for the differential model. Both equations show that the temperature profiles are similar in regions distant from the bed entrance, and that the temperatures decrease exponentially with axial distance. Here, $\lambda_1^{\,i}$ is a decreasing exponential function of i because λ_1 is a positive number less than unity (from the second property of **A**).

The radially averaged temperature, as discussed above, is the simple average of the radially averaged temperature of the conjugate models. Both averaging operations may be accomplished simultaneously by a row vector **V** which weights the temperature in each radial stage j in proportion to the volume of that stage. This averaging vector is

$$\mathbf{V} \equiv \frac{1}{2}\left[\frac{1}{M^2}\;\frac{4}{M^2}\;\frac{8}{M^2}\;\;\cdots\;\;\frac{4j-4}{M^2}\;\;\cdots\;\;\frac{4M-4}{M^2}\;\frac{2M-1}{M^2}\right] \tag{39}$$

The factor $\frac{1}{2}$ reflects the model averaging. The sum of the elements of **V** is, of course, unity. The radially averaged temperature at row i is

$$\overline{T}_i \equiv \mathbf{V}\mathbf{T}_i \tag{40}$$

Thus in the region of similar profiles, the radial average obtained from Equations (38) and (39) is

$$\overline{T}_i = \lambda_1^{\,i}\mathbf{V}\mathbf{J}(\lambda_1)\mathbf{T}_o/\varphi'(\lambda_1) \tag{41}$$

This averaging operation is the analog of the radial average performed by integration, Equation (15), in the differential representation.

The temperature of the fluid in the wall stages, necessary for the determination of the expression for H_1 from Equation (4), cannot easily be extracted from Equation (38). It can be readily obtained, however, from a total heat balance on row i. A total heat balance on row i follows from the appropriate addition of the original heat balances for each stage on row i, Equations (26), (27), (28) and (29). In terms of the radially averaged temperatures, this total heat balance is

$$\overline{T}_{i-1} - \overline{T}_i = \frac{4}{M}H_2 T_{wi} \tag{42}$$

where T_{wi} is the temperature of the fluid in the wall stages at row i

$$T_{wi} \equiv \frac{1}{2}\left[T_{i,M} + T_{i,M+1}\right] \tag{43}$$

Then by Equation (4), the effective coefficient H_1 is given by

$$H_1 = M(\overline{T}_{i-1} - \overline{T}_i)/(4\overline{T}_i) \tag{44}$$

Thus, in the region of similar profiles, Equations (44) and (41) give

$$H_1 = M(1-\lambda_1)/(4\lambda_1) \tag{45}$$

This equation expresses H_1 as an explicit function of H_2 and M. It is the finite stage analog of Equation (16).

Thus, the effective heat transfer coefficient H_1 for each model is a function of the dominant root of an equation characteristic of the heat balance for that model. This characteristic equation for the differential model is a transcendental equation [Equation (12)], and for the finite stage model it is a polynomial equation of degree $M + 1$

$$|\mathbf{A} - \lambda\mathbf{I}| = 0 \tag{46}$$

For the differential model, the dominant root β_1 is a function of one parameter only, Bi; but for the finite stage model, the dominant root λ_1 is a function of two parameters, H_2 and M, and M takes on only integer values. The root λ_1 of polynomial equation (46) can be found numerically by standard methods; however, because λ_1 is usually very close to unity, special care must be taken to find the difference between 1 and λ_1 accurately. Suitable numerical methods are discussed in Appendix A.†

The effective bed resistance may be calculated as before from Equation (5):

$$R(M, H_2) = \frac{4}{M}\left[\frac{\lambda_1(M, H_2)}{1-\lambda_1(M, H_2)}\right] - \frac{1}{H_2} \tag{47}$$

This is the finite stage analog of Equation (21) for the differential model.

Figure 6 shows R, determined from Equation (47), as a function of M and H_2. Although this graph of bed resistance has meaning only for integer values of M, continuous curves have been drawn through these points for clarity of presentation.

The bed resistances shown in this figure for the finite stage model resemble those of the differential model, shown in Figure 3. Quantitative comparison of the bed resistances of the two models, however, only leads to an estimate of the radial Peclet number of the finite stage model. This Peclet number for the finite stage model is around 8, but depends on just how it is defined (6, 7).

An analysis of the finite stage model for the case in which the heat flux at the wall is constant along the length

† See footnote on page 1015.

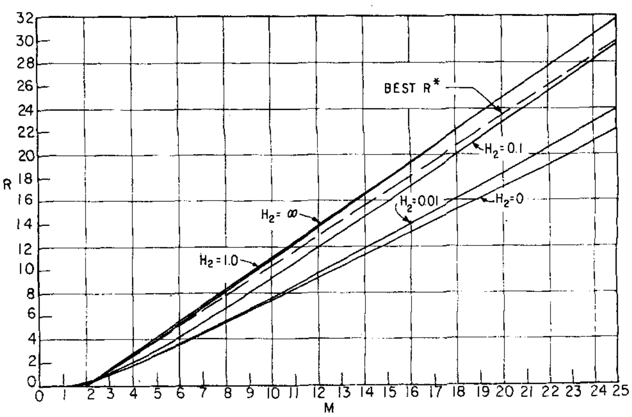

Fig. 6. The bed resistance R of the finite stage model as a function of M for selected values of H_2.

of the tube reveals that the bed resistance in this case is identical to the curve shown in Figure 6 for $H_2 = 0$. This is directly analogous to the situation shown earlier for the differential model. For the finite stage model, however, the shape of the radial temperature profiles has not been determined.

Because the bed resistances of the finite stage model are also only slightly dependent upon H_2, the bed resistance can be satisfactorily replaced with an approximate bed resistance R^*. The best choice for R^*, or the choice of R^* which yields the lowest maximum value of the error η defined in Equation (23), was determined by an analysis similar to that of the differential model. The details of this analysis are presented in Appendix *B*.† The best choice for R^* for the finite stage model is shown in Figure 6. This bed resistance is fitted (with an error in R^* of less than 0.2% for $M > 4$) by the line

$$R^* = 1.304\,M - 2.734 \tag{48}$$

With this bed resistance the maximum value of the error η is 6.8%.

CONCLUSIONS

For both the differential and the finite stage models, the local effective heat transfer coefficient in regions of similar temperature profiles is a simple function of a single dominant root of an equation characteristic of the heat balances of each model. Numerical values of this coefficient can therefore be determined by well-known methods of root finding.

A thermal resistance of the bed is defined as the difference between the total radial thermal resistance (the inverse of the effective coefficient) and the thermal resistance at the wall (the inverse of the wall heat transfer coefficient). For both models, this bed resistance is approximately a linear function of the bed diameter M and is not very sensitive to the wall heat transfer coefficient.

For each model an approximate bed resistance is found that is independent of the wall heat transfer coefficient and that can be used to estimate the total radial thermal resistance of the bed with an error of less than 7%. This approximate bed resistance is a linear function of M for each model.

The results of this study support the use of an effective local heat transfer coefficient of a packed bed in which the total radial thermal resistance of the bed is equal to the sum of a wall resistance and a bed resistance which depends only upon the properties of the fluid and the packing. This effective heat transfer coefficient can be used in one-dimensional representations of packed beds.

ACKNOWLEDGMENT

The authors acknowledge with thanks the support of this work by the U. S. Atomic Energy Commission through the Lawrence Radiation Laboratory; the National Science Foundation; and the Computer Center of the University of California, Berkeley.

NOTATION

$\mathbf{A}$ = matrix of coefficients of finite difference equations, defined by Equation (32)
Bi = Biot number, see Equation (13)
c = circumference of tube
c_p = heat capacity of the fluid
D_T = tube diameter
d_p = particle diameter
G = mass flow rate per unit cross-sectional void area
H_1, H_2 = dimensionless heat transfer coefficients, h_1/Gc_p and h_2/Gc_p, respectively
h_1 = effective wall heat transfer coefficient defined by Equation (1)
h_2 = heat transfer coefficient at the tube wall, as used in Equation (2)
$\mathbf{I}$ = the identity matrix
i, j = indices indicating row and column number, respectively. i is a discrete form of x, and j is a discrete form of $2r$
J_0, J_1 = Bessel functions of the first kind of order zero and one, respectively
$\mathbf{J}(\lambda)$ = $-\varphi(\lambda)[\mathbf{A} - \lambda\mathbf{I}]^{-1}$, the adjoint matrix of $[\mathbf{A} - \lambda\mathbf{I}]$
k_x, k_r = axial and radial effective thermal conductivities
M = number of particles across tube diameter D_T/d_p. M has only integer values in the finite stage model
Pe_x, Pe_r = axial and radial Peclet numbers, d_pGc_p/k_x and d_pGc_p/k_r, respectively
R = effective one-dimensional bed resistance defined by Equation (5)
R^* = an approximation to R
r = dimensionless radial position measured from tube axis in terms of d_p
T = normalized fluid temperature
$\overline{T}$ = normalized radially averaged temperature
T_o = normalized temperature of fluid entering bed
T_w = normalized temperature of the fluid nearest the wall of the tube
$\mathbf{T}_i$ = temperature vector defined by Equation (33)
$\mathbf{V}$ = row vector defined by Equation (39) which radially averages temperature in the finite stage model
x = dimensionless axial position measured from bed entrance in terms of d_p

Greek Letters

β_n = root of Equation (12)
β_1 = smallest positive root of Equation (12)
$\beta_{1\infty}^2$ = 5.783186 . . ., the square of the smallest positive root of $J_0(\beta) = 0$
δq = local rate of heat transfer through tube wall; δq is infinitesimal in differential model and is the finite flow through the tube wall of one wall stage in the finite stage model
η = error defined in Equation (23)
θ = fluid temperature
θ_c = temperature of medium external to tube
$\overline{\theta}$ = radially averaged fluid temperature
θ_o = temperature of fluid entering bed
λ = argument of $\varphi(\lambda)$, the characteristic polynomial of $\mathbf{A}$
λ_r = a root of $\varphi(\lambda) = 0$
λ_1 = largest positive root of $\varphi(\lambda) = 0$
$\varphi'(\lambda_r)$ = $d\varphi(\lambda)/d\lambda$ evaluated at $\lambda = \lambda_r$
$\varphi(\lambda)$ = $|\mathbf{A} - \lambda\mathbf{I}|$, the characteristic polynomial of $\mathbf{A}$
= negative root of the quadratic equation given with Equation (11)
ρ_1 = negative root associated with β_1

† See footnote on page 1015

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Manuscript received February 23, 1965; revision received June 11, 1965; paper accepted July 23, 1965.

Direct Contact Heat Transfer with Change of Phase: Condensation of Single Vapor Bubbles in an Immiscible Liquid Medium. Preliminary Studies

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Motion picture studies of condensation of isopentane bubbles rising in water elucidate the transfer mechanism involved in latent heat transport.

In general, two characteristic regions are noted. In the first region, up to some 80% liquid content, the bubbles deform and oscillate and heat is transferred by turbulent convection. In the second region the rate of transfer is controlled by the resistance of the condensed liquid, and heat is mainly by conduction. The effect of temperature differences between the bubble and the continuous phase (up to 3.5°C.) on the transfer coefficients could not be isolated, whereas that of the initial diameter of the bubble was quite marked.

The results are in agreement with those obtained in earlier studies of evaporating drops in immiscible liquids, indicating the similarity of the basic heat transfer mechanisms.

Condensation of vapors in immiscible liquid media as a means of latent heat transport has recently gained importance in connection with water desalination projects. This operation is usually coupled with an earlier stage of direct-contact evaporation, the two operations in series forming a closed thermodynamic cycle. In essence, this cycle is similar to the standard refrigeration cycles, with the important omission of the metallic heat transfer surface between the phases. The temperature level, as well as the fluid used as the dispersed phase, may, however, vary according to whether water vapor or ice is the (intermediate) product required. The advantages of this mode of operation over conventional heat transfer techniques are summarized in our earlier work on direct-contact evaporation (*1*, *2*).

Previous work (*3* to *6*) on condensation of vapors in immiscible liquid was aimed mainly at obtaining volumetric transfer coefficients in packed columns, in cocurrent pipe flow, and in simulated spray columns. Condensation of isobutane in an ice-packed column was reported by Wiegandt (*3*). Heat transfer coefficients per column cross section were reported to vary from 5,000 to 3,300 kcal./(hr.) (°C.) (sq. meter) with a temperature difference from 1° to 3°C. Condensation studies in water-ice slurries in a 2-in. column indicated that condensation rates increased with increasing ice content in the slurry. With butane feed of 1.24 lb./sq.ft. (column cross section) and an average temperature difference of 3°C., the overall heat transfer coefficient was reported as approximately 10,000 kcal./(hr.) (°C.) (sq. meter).

Condensation of steam in Aroclor in countercurrent packed beds and in cocurrent flow in a pipe using a Venturi as a mixing device was reported by Wilke (*4*) and Lackey (*5*). For the packed-bed column experimental H.T.U. (height of transfer unit, liquid controlling resistance) varied between about 25 and 50 cm. for Aroclor flow rates of about 7.5×10^4 to 15×10^4 kg./

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